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ARTICLE

Convergence of the `QuickVal` residual

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Abstract

`QuickSelect` (also known as `Find`), introduced by Hoare ((1961) *Commun. ACM* **4** 321–322.), is a randomized algorithm for selecting a specified order statistic from an input sequence of n objects, or rather their identifying labels usually known as *keys*. The keys can be numeric or symbol strings, or indeed any labels drawn from a given linearly ordered set. We discuss various ways in which the cost of comparing two keys can be measured, and we can measure the efficiency of the algorithm by the total cost of such comparisons.

We define and discuss a closely related algorithm known as `QuickVal` and a natural probabilistic model for the input to this algorithm; `QuickVal` searches (almost surely unsuccessfully) for a specified *population* quantile $\alpha \in [0, 1]$ in an input sample of size n. Call the total cost of comparisons for this algorithm S_n. We discuss a natural way to define the random variables $S_1, S_2, \ldots$ on a common probability space. For a general class of cost functions, Fill and Nakama ((2013) *Adv. Appl. Probab.* **45** 425–450.) proved under mild assumptions that the scaled cost S_n/n of `QuickVal` converges in L^p and almost surely to a limit random variable S. For a general cost function, we consider what we term the `QuickVal` residual:

$$\rho_n := \frac{S_n}{n} - S.$$

The residual is of natural interest, especially in light of the previous analogous work on the sorting algorithm `QuickSort` (Bindjeme and Fill (2012) *23rd International Meeting on Probabilistic, Combinatorial, and Asymptotic Methods for the Analysis of Algorithms (AofA'12), Discrete Mathematics, and Theoretical Computer Science Proceedings, AQ*, Association: Discrete Mathematics and Theoretical Computer Science, Nancy, pp. 339–348; Neininger (2015) *Random Struct. Algorithms* **46** 346–361; Fuchs (2015) *Random Struct. Algorithms* **46** 677–687; Grübel and Kabluchko (2016) *Ann. Appl. Probab.* **26** 3659–3698; Sulzbach (2017) *Random Struct. Algorithms* **50** 493–508). In the case $\alpha = 0$ of `QuickMin` with unit cost per key-comparison, we are able to calculate–àla Bindjeme and Fill for `QuickSort` (Bindjeme and Fill (2012) *23rd International Meeting on Probabilistic, Combinatorial, and Asymptotic Methods for the Analysis of Algorithms (AofA'12), Discrete Mathematics and Theoretical Computer Science Proceedings*, AQ, Association: Discrete Mathematics and Theoretical Computer Science, Nancy, pp. 339–348.)–the exact (and asymptotic) L^2-norm of the residual. We take the result as motivation for the scaling factor $\sqrt{n}$ for the `QuickVal` residual for *general* population quantiles and for *general* cost. We then prove *in general* (under mild conditions on the cost function) that $\sqrt{n}\,\rho_n$ converges in law to a scale mixture of centered Gaussians, and we also prove convergence of moments.

Keywords: QuickSelect; Find; QuickQuant; QuickVal residual; natural coupling; symbol comparisons; key comparisons; probabilistic source; tameness

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1. Introduction

Parts of Sections 1–2 are repeated nearly verbatim, for the convenience of the reader and with permission of the publisher, from [11]. Note, however, that we have updated the literature review, perhaps most notably including an excellent sequel to this paper, namely, [20]; see especially Remark 1.2.

In this section, we describe QuickSelect and QuickQuant and give historical background. The main result of this paper, Theorem 5.1, concerns an algorithm very closely related to QuickQuant known as QuickVal, which is described in Section 4. In Section 3 we will define and consider the algorithm QuickMin, which can be viewed a special case of either QuickQuant or QuickVal.

QuickSelect (also known as FIND), introduced by Hoare [17], is a randomized algorithm (a close cousin of the randomized sorting algorithm QuickSort, also introduced by Hoare [18]) for selecting a specified order statistic from an input sequence of objects, or rather their identifying labels usually known as *keys*. The keys can be numeric or symbol strings, or indeed any labels drawn from a given linearly ordered set. Suppose we are given keys $y_1, \ldots, y_n$ and we want to find the mth smallest among them. The algorithm first selects a key (called the pivot) uniformly at random. It then compares every other key to the pivot, thereby determining the rank, call it r, of the pivot among the n keys. If $r = m$, then the algorithm terminates, returning the pivot key as output. If $r > m$, then the algorithm is applied recursively to the keys smaller than the pivot to find the mth smallest among those; while if $r < m$, then the algorithm is applied recursively to the keys larger than the pivot to find the $(m - r)$th smallest among those. More formal descriptions of QuickSelect can be found in [17] and [22], for example.

The cost of running QuickSelect can be measured (somewhat crudely) by assessing the cost of comparing keys. We assume that every comparison of two (distinct) keys costs some amount that is perhaps dependent on the values of the keys, and then the cost of the algorithm is the sum of the comparison costs.

Historically, it was customary to assign unit cost to each comparison of two keys, irrespective of their values. We denote the (random) key-comparisons-count cost for QuickSelect by $K_{n,m}$. There have been many studies of the random variables $K_{n,m}$, including [3, 24, 15, 23, 14, 4, 19, 5, 9, 10]. But unit cost is not always a reasonable model for comparing two keys. For example, if each key is a string of symbols, then a more realistic model for the cost of comparing two keys is the value of the first index at which the two symbol strings differ. To date, only a few papers [29, 6, 7, 20] have considered QuickSelect from this more realistic symbol-comparisons perspective. As in [7] and [11], in this paper we will treat a rather general class of cost functions that includes both key-comparisons cost and symbol-comparisons cost.

In our set-up (to be described in detail in Section 2) for this paper, we will consider a variety of probabilistic models (called *probabilistic sources*) for how a key is generated as an infinite-length string of symbols, but we will always assume that the keys form an infinite sequence of independent and identically distributed and almost surely distinct symbol strings. This gives us, on a single probability space, all the randomness needed to run QuickSelect for *every* value of n and *every* value of $m \in \{1, \ldots, n\}$ by always choosing the *first* key in the sequence as the pivot (and maintaining initial relative order of keys when the algorithm is applied recursively); this is what is meant by the *natural coupling* (cf. [8, Section 1]) of the runs of the algorithm for varying n and m (and varying cost functions).

When considering asymptotics of the cost of QuickSelect as the number of keys tends to ∞, it becomes necessary to let the order statistic m_n depend on the number of keys n. When $m_n/n \to \alpha \in [0, 1]$, we refer to QuickSelect for finding the m_nth order statistic among n keys as QuickQuant(n, α). As explained in [8, Section 1], the natural coupling allows us to consider stronger forms of convergence for the cost of QuickQuant(n, α) than convergence in distribution, such as almost sure convergence and convergence in L^p. Fill and Nakama [7] prove, under certain

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'tameness' conditions (to be reviewed later) on the probabilistic source and the cost function, that, for each fixed α, the cost of $\texttt{QuickQuant}(n, \alpha)$, when scaled by n, converges both in L^p and almost surely to a limiting random variable. Fill and Matterer [11] extend these univariate convergence results to results about convergence of certain related stochastic processes.

Closely related to $\texttt{QuickQuant}(n, \alpha)$ is an algorithm called $\texttt{QuickVal}(n, \alpha)$, detailed in Section 4. Employing the natural coupling, $\texttt{QuickVal}(n, \alpha)$ searches (almost surely unsuccessfully) for a specified *population* quantile $\alpha \in [0, 1]$ in an input sample of size n. Call the total cost of comparisons for this algorithm S_n. For a general class of cost functions, Fill and Nakama [7] proved under mild assumptions that the scaled cost S_n/n of QuickVal converges in L^p and almost surely to a limit random variable S. For a general cost function, we consider what we term the QuickVal residual:

$$\rho_n := \frac{S_n}{n} - S. \tag{1.1}$$

The residual is of natural interest, especially in light of the previous analogous work on QuickSort [1, 26, 13, 16, 28].

An outline for this paper is as follows. First, in Section 2 we carefully describe our set-up and, in some detail, discuss probabilistic sources, cost functions, and tameness; we also discuss the idea of *seeds*, which allow us a unified treatment of all sources. Section 3 concerns QuickMin (the case $\alpha = 0$ of QuickVal) with unit cost per key-comparison, for which we are able to calculate – à la Bindjeme and Fill for QuickSort [1] – the exact (and asymptotic) L^2-norm of the residual; the result is Theorem 3.1, which we take as motivation for the scaling factor $\sqrt{n}$ for the QuickVal residual for *general* population quantiles and for *general* cost. The remainder of the paper is devoted to establishing convergence of the QuickVal residual. Section 4 introduces notation needed to state the main theorem and establishes an important preliminary result (Lemma 4.2). We state and prove the main theorem (Theorem 5.1), which asserts that the scaled cost of the QuickVal residual converges in law to a scale mixture of centered Gaussians, in Section 5; and in Section 6 we prove the corresponding convergence of moments.

Remark 1.1. *As recalled from* [7] *at the end of our Section* 2.1, *many common sources, including memoryless and Markov sources, have the property that the source-specific cost function β corresponding to the symbol-comparisons cost for comparing keys is ϵ-tame for every $\epsilon > 0$. Thus our main result, Theorem* 5.1, *applies to all such sources.*

Remark 1.2. *In very recent work, Ischebeck and Neininger* [20] *extend our main Theorem* 5.1 *from univariate normal convergence for each α to Gaussian-process convergence, treating α as a parameter, in the metric space of càdlàg functions endowed with the Skorokhod metric.*

To motivate the reader, here is a fairly easily understood instance of our main Theorem 5.1. Suppose that keys arrive as i.i.d. uniform(0, 1) random variables, and suppose that cost is measured classically as the number of key comparisons. Using QuickVal to search for population quantile $\alpha \in [0, 1]$, suppose for each $k \geq 0$ that the search has been narrowed to the (random) interval (L_k, R_k) after k steps of the algorithm have been carried out; in particular, $L_0 = 0$ and $R_0 = 1$. Let $I_k := R_k - L_k$. Then, as shown in the proof of Theorem 5.1, the random series (of positive terms) in the expressions

$$\begin{aligned} \sigma_\infty^2 &= \sum_{k=1}^{\infty} (I_{k-1} - I_{k-1}^2) + 2 \sum_{\ell=1}^{\infty} \sum_{k=1}^{\ell-1} (I_{\ell-1} - I_{k-1} I_{\ell-1}) \\ &= \sum_{k=1}^{\infty} I_k(1 - I_k) + 2 \sum_{\ell=2}^{\infty} I_\ell \sum_{k=1}^{\ell-1} (1 - I_k) \end{aligned}$$

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converge with probability one, and the residual ρ_n given by (1.1) converges in distribution to $\sigma_\infty Z$, where Z has a standard normal distribution and is independent of σ_∞.

2. Set-up

2.1 Probabilistic sources

Let us define the fundamental probabilistic structure underlying the analysis of `QuickSelect`. We assume that keys arrive independently and with the same distribution and that each key is composed of a sequence of symbols from some finite or countably infinite alphabet. Let Σ be this alphabet (which we assume is totally ordered by $\leq$). Then a key is an element of Σ^∞ [ordered by the lexicographic order, call it $\preceq$, corresponding to $(\Sigma, \leq)$] and a *probabilistic source* is a stochastic process $W = (W_1, W_2, W_3, \ldots)$ such that for each i the random variable W_i takes values in Σ. We will impose restrictions on the distribution of W that will have as a consequence that (with probability one) all keys are distinct.

We denote the cost (assumed to be non-negative) of comparing two keys w, w' by $\operatorname{cost}(w, w')$. As two examples, the choice $\operatorname{cost}(w, w') \equiv 1$ gives rise to a key-comparisons analysis, whereas if words are symbol strings then a symbol-comparisons analysis is obtained by letting $\operatorname{cost}(w, w')$ be the first index at which w and w' disagree.

Since Σ^∞ is totally ordered, a probabilistic source W is governed by a distribution function F defined for $w \in \Sigma^\infty$ by

$$F(w) := \mathbb{P}(W \preceq w).$$

Then the corresponding inverse probability transform M, defined by

$$M(u) := \inf \left\{ w \in \Sigma^\infty : u \leq F(w) \right\},$$

has the property that if $U \sim \operatorname{uniform}(0, 1)$, then $M(U)$ has the same distribution as W. We refer to such uniform random variables U as *seeds*.

Using this technique, we can define a *source-specific cost function*

$$\beta : (0, 1) \times (0, 1) \to [0, \infty)$$

by $\beta(u, v) := \operatorname{cost}(M(u), M(v))$.

Definition 2.1. *Let $0 < c < \infty$ and $0 < \epsilon < \infty$. A source-specific cost function β is said to be (c, ϵ)-tame if for $0 < u < t < 1$, we have*

$$\beta(u, t) \leq c\,(t - u)^{-\epsilon},$$

and is said to be ϵ-tame if it is (c, ϵ)-tame for some c.

For further important background on sources, cost functions, and tameness, we refer the reader to Section 2.1 (see especially Definitions 2.3–2.4 and Remark 2.5) in Fill and Nakama [7]. Note in particular that many common sources, including memoryless and Markov sources, have the property that the source-specific cost function β corresponding to symbol-comparisons cost for comparing keys is ϵ-tame for every $\epsilon > 0$.

2.2 Tree of seeds and the `QuickSelect` tree processes

Let $\mathcal{T}$ be the collection of (finite or infinite) rooted ordered binary trees (whenever we refer to a binary tree we will assume it is of this variety) and let $\overline{T} \in \mathcal{T}$ be the complete infinite binary tree. We will label each node θ in a given tree $T \in \mathcal{T}$ by a binary sequence representing the path from the root to θ, where 0 corresponds to taking the left child and 1 to taking the right. We consider the set of real-valued stochastic processes each with index set equal to some $T \in \mathcal{T}$. For such a

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process, we extend the index set to $\overline{T}$ by defining $X_\theta = 0$ for $\theta \in \overline{T} \setminus T$. We will have need for the following definition of levels of a binary tree.

Definition 2.2. *For $0 \le k < \infty$, we define the k^{th} level Λ_k of a binary tree as the collection of vertices that are at distance k from the root.*

Let $\Theta = \bigcup_{0 \le k < \infty} \{0, 1\}^k$ be the set of all finite-length binary strings, where $\{0, 1\}^0 = \{\varepsilon\}$ with ε denoting the empty string. Set $L_\varepsilon := 0$, $R_\varepsilon := 1$, and $\tau_\varepsilon := 1$. Then, for $\theta \in \Theta$, we define $|\theta|$ to be the length of the string θ, and $\upsilon_\theta(n)$ to be the size (through the arrival of the n^{th} key) of the subtree rooted at node θ. Given a sequence of independent and identically distributed (iid) seeds $U_1, U_2, U_3, \ldots$, we recursively define

$$\begin{aligned} \tau_\theta &:= \inf\{i\colon L_\theta < U_i < R_\theta\}, \\ L_{\theta 0} &:= L_\theta,\ L_{\theta 1} := U_{\tau_\theta}, \\ R_{\theta 0} &:= U_{\tau_\theta},\ R_{\theta 1} := R_\theta, \end{aligned}$$

where $\theta_1\theta_2$ denotes the concatenation of $\theta_1, \theta_2 \in \Theta$. For a source-specific cost function β and $0 \le p < \infty$, we define

$$\begin{aligned} S_{n,\theta} &:= \sum_{\tau_\theta < i \le n} \mathbf{1}(L_\theta < U_i < R_\theta)\beta(U_i, U_{\tau_\theta}), \\ I_p(x, a, b) &:= \int_a^b \beta^p(u, x)\, du, \\ I_{p,\theta} &:= I_p(U_{\tau_\theta}, L_\theta, R_\theta), \\ I_\theta &:= I_{1,\theta}, \\ C_\theta &:= (\tau_\theta, U_{\tau_\theta}, L_\theta, R_\theta). \end{aligned}$$

In some later definitions we will make use of the positive part function defined as usual by $x^+ := x\,\mathbf{1}(x > 0)$. Given a source-specific cost function β and the seeds $U_1, U_2, U_3, \ldots$, we define the nth `QuickSelect` seed process as the n-nodes binary tree indexed stochastic process obtained by successive insertions of $U_1, \ldots, U_n$ into an initially empty binary search tree (BST).

Before we use these random variables, we supply some understanding of them for the reader. The arrival time τ_θ is the index of the seed that is slotted into node θ in the construction of the `QuickSelect` seed process. Note that for each $\theta \in \Theta$ we have $P(\tau_\theta < \infty) = 1$. The interval (L_θ, R_θ) provides sharp bounds for all seeds arriving after time τ_θ that interact with U_{τ_θ} in the sense of being placed in the subtree rooted at U_{τ_θ}. A crucial observation is that, conditioned on C_θ, the sequence of seeds $U_{\tau_\theta+1}, U_{\tau_\theta+2}, \ldots$ are iid uniform(0, 1); thus, again conditioned on C_θ, the sum $S_{n,\theta}$ is the sum of $(n - \tau_\theta)^+$ iid random variables. Note that when $n \le \tau_\theta$ the sum defining $S_{n,\theta}$ is empty and so $S_{n,\theta} = 0$; in this case, we shall conveniently interpret $S_{n,\theta}/(n - \tau_\theta)^+ = 0/0$ as 0. The random variable $S_{n,\theta}$ is the total cost of comparing the key with seed U_{τ_θ} with keys (among the first n overall to arrive) whose seeds fall in the interval (L_θ, R_θ), and $I_{p,\theta}$ is the conditional pth moment of the cost of one such comparison: If we let $U \sim \text{uniform}(0, 1)$ independent of C_θ, then

$$I_{p,\theta} = \mathbb{E}\left[\, \mathbf{1}(L_\theta < U < R_\theta)\beta^p(U, U_{\tau_\theta}) \middle|\, C_\theta \right].$$

Conditioned on C_θ, the term $S_{n,\theta}$ is the sum of $(n - \tau_\theta)^+$ iid random variables with pth moment $I_{p,\theta}$.

We define the n^{th} `QuickSelect` *tree process* as the binary-tree-indexed stochastic process $S_n = (S_{n,\theta})_{\theta \in \Theta}$ and the *limit* `QuickSelect` *tree process* (so called in light of [11, Master Theorem 4.1]) by $I = (I_\theta)_{\theta \in \Theta}$.

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We recall from [11] an easily established lemma that will be invoked in Remark 4.4 and in the proof of Lemma 6.3.

Lemma 2.3 (Lemma 3.1 of [11]). *If β is (c,ϵ)-tame with $0 \le \epsilon < 1/s$, then for each fixed node $\theta \in \Lambda_k$ and $0 \le r < \infty$ we have*

$$\mathbb{E}I_{s,\theta}^r \le \left(\frac{2^{s\epsilon}c^s}{1-s\epsilon}\right)^r \left(\frac{1}{r+1-rs\epsilon}\right)^k.$$

3. Exact L^2 asymptotics for `QuickMin` residual

Before deriving a limit law for `QuickVal` under general source-specific cost functions β, we motivate the scaling factor of $\sqrt{n}$ in Theorem 5.1. We consider the case of `QuickMin` (i.e., `QuickSelect` for the minimum key) with key-comparisons cost ($\beta \equiv 1$). Note that the operation of `QuickMin` and `QuickVal` with $\alpha = 0$ are identical. Our goal in this section is to establish Theorem 3.1, which gives exact and asymptotic expansions for the second moment of the residual in this special case.

Let K_n denote the key-comparisons cost of `QuickMin` and define

$$Y_n := \frac{K_n - \mu_n}{n+1}, \tag{3.1}$$

where $\mu_n := \mathbb{E}K_n = 2(n - H_n)$ for each n [22]. A consequence of [19, Theorem 1] is that $K_n/n \xrightarrow{\mathcal{L}} D$, where $D \stackrel{\mathcal{L}}{=} \sum_{k=0}^{\infty}\prod_{j=0}^{k} U_j$ has a Dickman distribution [19], with $\mu := \mathbb{E}D = 2$ (here $U_0 := 1$). (Note that [19] refers to $D-1$ as having a Dickman distribution; we ignore this distinction.) Applying [7, Theorems 3.1 and 3.2] to the special case of `QuickMin` using key-comparisons costs yields the stronger result that Y_n converges to a limit random variable Y in L^p for any $p \ge 1$ and almost surely. We can then set

$$D := Y + \mu = Y + 2, \tag{3.2}$$

and this D has a Dickman distribution as defined above.

The main result of this Section 3 is the exact calculation of of the second moment of $Y_n - Y$:

Theorem 3.1. *For Y_n and Y defined previously, we have*

$$\begin{aligned} a_n^2 := \mathbb{E}(Y_n - Y)^2 &= (n+1)^{-2}\left[\frac{3}{2}n + 4H_n - 4H_n^{(2)} + \frac{1}{2}\right] \\ &= \frac{3}{2}n^{-1} + O\left(\frac{\log n}{n^2}\right). \end{aligned}$$

The remainder of this section builds to the proof of Theorem 3.1. Define

$$N_n := \#\{1 < i \le n : U_i < U_1\} \tag{3.3}$$

(i.e., the number of keys that fall into the left subtree of the `QuickSelect` seed process). To begin the derivation, note that $K_n = n - 1 + \widetilde{K}_{N_n}$, where $\widetilde{K}_{N_n}$ is the key-comparisons cost for `QuickMin` applied to the left subtree. Note also that the same equation holds as equality in law if the process $\widetilde{K}$ has the same distribution as the process K and is independent of N_n. We also have $D = 1 + U\widetilde{D}$ with $U := U_1$ and $\widetilde{D} \stackrel{\mathcal{L}}{=} D$ independent.

Make the following definitions:

$$\begin{aligned} Y_{n,0} &:= \frac{K_{N_n} - \mu_{N_n}}{N_n + 1}, \\ Y^{(0)} &:= \widetilde{D} - \mu. \end{aligned} \tag{3.4}$$

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Then we can express the residual $Y_n - Y$ in terms of these "smaller versions" of Y_n and Y:

$$\begin{aligned} Y_n - Y &= \frac{n-1+K_{N_n}-\mu_n}{n+1} - \left(1+\ U\widetilde{D}-\mu\right) \\ &= \left(\frac{N_n+1}{n+1}\right) Y_{n,0} - UY^{(0)} + \frac{n-1}{n+1} - 1 + \frac{\mu_{N_n}-\mu_n}{n+1} - U\mu + \mu \\ &= \left(\frac{N_n+1}{n+1}\right) Y_{n,0} - UY^{(0)} + C_n(N_n)\frac{n}{n+1} - C(U), \end{aligned} \tag{3.5}$$

where $C_n(i) := n^{-1}(n-1+\mu_i-\mu_n)$ and $C(x) := \mu x - 1 = 2x - 1$. Observe that with these definitions, we can break up the previous equation as

$$Y_n - Y = W_1 + W_2, \tag{3.6}$$

where

$$W_1 := \frac{N_n+1}{n+1} Y_{n,0} - UY^{(0)}, \quad W_2 := C_n(N_n)\frac{n}{n+1} - C(U).$$

Conditionally given N_n and U, the random variable W_2 is constant and W_1 has mean zero, so

$$a_n^2 = \mathbb{E}\,(Y_n - Y)^2 = \mathbb{E}W_1^2 + \mathbb{E}W_2^2.$$

Consider the first term $\mathbb{E}W_1^2$.

Lemma 3.2.

$$\mathbb{E}W_1^2 = \frac{1}{n}\sum_{k=0}^{n-1}\frac{(k+1)^2}{(n+1)^2}a_k^2 + \frac{1}{12(n+1)}.$$

Proof. If we define

$$Z_1 := \frac{N_n+1}{n+1}\left(Y_{n,0} - Y^{(0)}\right), \quad Z_2 := \left(\frac{N_n+1}{n+1} - U\right) Y^{(0)},$$

then $W_1 = Z_1 + Z_2$ and so $\mathbb{E}W_1^2 = \mathbb{E}Z_1^2 + \mathbb{E}Z_2^2 + 2\mathbb{E}(Z_1Z_2)$.

For the cross term $\mathbb{E}(Z_1Z_2)$, conditionally given N_n the random variable U is distributed $\mathrm{Beta}(N_n+1, n-N_n)$. Therefore,

$$\mathbb{E}(Z_1Z_2) = \mathbb{E}\left\{\mathbb{E}\left[Z_1\left(\frac{N_n+1}{n+1} - U\right)Y^{(0)}\,\middle|\, N_n, Y_{n,0}, Y^{(0)}\right]\right\} = 0.$$

Next consider the term $\mathbb{E}Z_1^2$.

Remark 3.3. *The conditional joint distribution of the process $(Y_{n,0})_{n\geq 0}$ and the random variable $Y^{(0)}$ given N_n is the conditional joint distribution of the process $(Y^*_{N_n})_{n\geq 0}$ and the random variable Y^* given N_n, where the process (Y^*_n) and the random variable Y^* are independent of N_n and have (unconditionally) the same joint distribution as the process (Y_n) and the random variable Y.*

In light of the preceding remark,

$$\mathbb{E}Z_1^2 = \mathbb{E}\left[\left(\frac{N_n+1}{n+1}\right)^2 \left(Y^*_{N_n} - Y^*\right)^2\right].$$

Since $N_n \sim \mathrm{unif}\,\{0, 1, 2, \ldots, n-1\}$, conditioning on N_n gives

$$\mathbb{E}Z_1^2 = \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k+1}{n+1}\right)^2 a_k^2.$$

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Finally, consider the term $\mathbb{E}Z_2^2$. Since $Y^{(0)}$ is independent of N_n and U, we have

$$\mathbb{E}Z_2^2 = \mathbb{E}{Y^{(0)}}^2\, \mathbb{E}\left(\frac{N_n+1}{n+1} - U\right)^2.$$

Recall that $N_n \sim \text{unif}\,\{0, 1, \ldots, n-1\}$ and that conditionally given N_n, we have $U \sim \text{Beta}(N_n + 1, n - N_n)$; therefore,

$$\mathbb{E}\left(\frac{N_n+1}{n+1} - U\right)^2 = \frac{1}{n}\sum_{k=0}^{n-1} \frac{(k+1)(n-k)}{(n+1)^2(n+2)} = \frac{1}{6(n+1)}. \tag{3.7}$$

Since $\mathbb{E}{Y^{(0)}}^2 = 1/2$ [21], we have that

$$\mathbb{E}Z_2^2 = \frac{1}{12(n+1)}.$$

Putting these calculations together, we get that

$$\mathbb{E}W_1^2 = \frac{1}{n}\sum_{k=0}^{n-1} \frac{(k+1)^2}{(n+1)^2} a_k^2 + \frac{1}{12(n+1)}.$$

□

Now we consider the term $\mathbb{E}W_2^2$.

Lemma 3.4.

$$\mathbb{E}W_2^2 = \frac{2}{3(n+1)} + \left(\frac{2}{n+1}\right)^2 \left(1 - \frac{H_n}{n}\right).$$

Proof. We have

$$\begin{aligned}
W_2 &= C_n(N_n)\frac{n}{n+1} - C(U)\\
&= \frac{n}{n+1}\left[\frac{1}{n}\left(n - 1 + 2(N_n - H_{N_n}) - 2(n - H_n)\right)\right] + 1 - 2U\\
&= \frac{1}{n+1}\left[2(N_n - H_{N_n}) - n + 2H_n - 1\right] + 1 - 2U\\
&= 2\left(\frac{N_n+1}{n+1} - U\right) - \frac{2}{n+1} + \frac{1}{n+1}\left(-n + 2H_n - 2H_{N_n} - 1\right) + 1\\
&= 2\left(\frac{N_n+1}{n+1} - U\right) + \frac{2}{n+1}\left(H_n - H_{N_n} - 1\right).
\end{aligned} \tag{3.8}$$

Squaring and then taking expectations, we find

$$\begin{aligned}
\mathbb{E}W_2^2 =& 4\mathbb{E}\left(\frac{N_n+1}{n+1} - U\right)^2 + \mathbb{E}\left[\left(\frac{2}{n+1}\right)^2 (H_n - H_{N_n} - 1)^2\right]\\
&+ 4\mathbb{E}\left[\left(\frac{N_n+1}{n+1} - U\right)\left(\frac{2}{n+1}(H_n - H_{N_n} - 1)\right)\right].
\end{aligned}$$

Recall that conditionally given N_n we have $U \sim \text{Beta}(N_n + 1, n - N_n)$, which implies that the cross term vanishes; therefore, it suffices to consider the two squared terms. From (3.7) we know that

$$4\mathbb{E}\left(\frac{N_n+1}{n+1} - U\right)^2 = \frac{2}{3(n+1)}.$$

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We now proceed to treat the final term

$$L_n := \mathbb{E}\left[\left(\frac{2}{n+1}\right)^2 \left(H_n - H_{N_n} - 1\right)^2\right]$$
$$= \left(\frac{2}{n+1}\right)^2 \left[\mathbb{E}(H_n - H_{N_n})^2 - 2\mathbb{E}(H_n - H_{N_n}) + 1\right]. \tag{3.9}$$

We can compute the first and second moments of $H_n - H_{N_n}$ as follows. Fixing n, for $1 \le i \le n$ consider the events $B_i := \{N_n < i\}$, which satisfy $\mathbb{P}(B_i) = \frac{i}{n}$ and $B_i \cap B_j = B_{i\wedge j}$ (where $\wedge$ denotes minimum). Note that

$$H_n - H_{N_n} = \sum_{i=N_n+1}^{n} \frac{1}{i} = \sum_{i=1}^{n} \frac{\mathbf{1}(B_i)}{i}$$

Thus

$$\mathbb{E}(H_n - H_{N_n}) = \sum_{i=1}^{n} \frac{\mathbb{P}(B_i)}{i} = \sum_{i=1}^{n} \frac{1}{n} = 1$$

and

$$\mathbb{E}(H_n - H_{N_n})^2 = \mathbb{E}\left[\sum_{i=1}^{n} \frac{\mathbf{1}(B_i)}{i}\right]^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbb{P}(B_i \cap B_j)}{ij} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{i \wedge j}{ij}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} \frac{j}{ij} - \frac{1}{n}\sum_{i=1}^{n} \frac{i}{i^2} = 2 - \frac{H_n}{n}.$$

Therefore we get that

$$\mathbb{E}W_2^2 = \frac{2}{3(n+1)} + \left(\frac{2}{n+1}\right)^2 \left(1 - \frac{H_n}{n}\right),$$

as desired. □

We will also need the following well-known (and very easy to derive) solution to a "divide-and-conquer" recurrence in the proof of Theorem 3.1.

Lemma 3.5. *Let $(A_n)_{n\ge 0}$ and $(B_n)_{n\ge 1}$ be sequences of real numbers that satisfy*

$$A_n = \frac{1}{n}\sum_{k=0}^{n-1} A_k + B_n,$$

for $n \ge 1$. Then for $n \ge 0$ we have

$$A_n = A_0 + B_n + \sum_{k=1}^{n-1} \frac{1}{k+1} B_k, \tag{3.10}$$

with $B_0 := 0$.

Proof of Theorem 3.1. Combining the expressions for $\mathbb{E}W_1^2$ and $\mathbb{E}W_2^2$ gives

$$(n+1)^2 a_n^2 = \frac{1}{n}\sum_{k=0}^{n-1} (k+1)^2 a_k^2 + \frac{n+1}{12} + \frac{2(n+1)}{3} + 4\left(1 - \frac{1}{n} H_n\right). \tag{3.11}$$

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If we define

$$b_n := \frac{n+1}{12} + \frac{2(n+1)}{3} + 4\left(1 - \frac{1}{n}H_n\right) = \frac{3(n+1)}{4} + 4\left(1 - \frac{1}{n}H_n\right),$$

then Lemma 3.5 implies

$$(n+1)^2 a_n^2 = a_0^2 + \sum_{j=1}^{n-1} \frac{b_j}{j+1} + b_n.$$

Plugging in b_n gives

$$(n+1)^2 a_n^2 = a_0^2 + \sum_{j=1}^{n-1} \left[\frac{3(j+1)}{4(j+1)} + \frac{4}{j+1}\left(1 - \frac{1}{j}H_j\right)\right] + \frac{3(n+1)}{4} + 4\left(1 - \frac{1}{n}H_n\right).$$

Simplifying this expression gives

$$\begin{aligned}
(n+1)^2 a_n^2 &= a_0^2 + \frac{3}{4}(n-1) + \sum_{j=1}^{n-1} \frac{4}{j+1} - \sum_{j=1}^{n-1} \frac{4}{j(j+1)}H_j + \frac{19}{4} + \frac{3n}{4} - \frac{4}{n}H_n \\
&= \frac{1}{2} + \frac{3}{2}n + 4H_n - \frac{4}{n}H_n - \sum_{j=1}^{n-1} \frac{4}{j(j+1)}H_j \\
&= \frac{1}{2} + \frac{3}{2}n + 4H_n - \frac{4}{n}H_n - 4\left(H_n^{(2)} - \frac{H_n}{n}\right) \\
&= \frac{3}{2}n + 4H_n - 4H_n^{(2)} + \frac{1}{2},
\end{aligned}$$

where $a_0^2 = 1/2$ was substituted in the second equality. Therefore, we can conclude that

$$\begin{aligned}
a_n^2 = \mathbb{E}(Y_n - Y)^2 &= (n+1)^{-2}\left[\frac{3}{2}n + 4H_n - 4H_n^{(2)} + \frac{1}{2}\right] \\
&= \frac{3}{2}n^{-1} + O\left(\frac{\log n}{n^2}\right).
\end{aligned}$$

□

4. `QuickVal` and mixing distribution for residual limit distribution

Our main theorem, Theorem 5.1, asserts that the scaled `QuickVal` residual cost converges in law to a scale mixture of centered Gaussians. In this section, we introduce needed notation and prove Lemma 4.2, which gives an explicit representation of the mixing distribution.

Consider a BST constructed by the insertion (in order) of the n seeds. Then `QuickQuant`(n, α) follows the path from the root to the node storing the m_n^{th} smallest key, where $m_n/n \to \alpha$.

For `QuickVal`(n, α), consider the same BST of seeds with the additional value α inserted (last). Then `QuickVal`(n, α) follows the path from the root to this α-node. Almost surely for n large and k fixed, the difference between these two algorithms in costs associated with the k-th pivot is negligible to lead order [7, (4.2)]. See [29] or [7] for a more complete description.

When considering `QuickVal`, we will simplify the notation since we will only need to reference one path of nodes from the root to a leaf in the `QuickSelect` process tree. For this we define similar notation indexed by the pivot index (i.e., by the level in the tree). Set $L_0 := 0$, $R_0 := 1$, and

$\tau_0 := 1$. Then, for $k \geq 1$, we define

$$\tau_k := \inf\{i\colon L_{k-1} < U_i < R_{k-1}\}, \tag{4.1}$$

$$L_k := \mathbf{1}(U_{\tau_k} < \alpha)U_{\tau_k} + \mathbf{1}(U_{\tau_k} > \alpha)L_{k-1}, \tag{4.2}$$

$$R_k := \mathbf{1}(U_{\tau_k} < \alpha)R_{k-1} + \mathbf{1}(U_{\tau_k} > \alpha)U_{\tau_k}, \tag{4.3}$$

$$C_k := (L_{k-1}, R_{k-1}, \tau_k, U_{\tau_k}) \tag{4.4}$$

$$X_{k,i} := \mathbf{1}(L_{k-1} < U_i < R_{k-1})\beta(U_i, U_{\tau_k}), \tag{4.5}$$

$$S_{k,n} := \sum_{i\colon \tau_k < i \leq n} X_{k,i}. \tag{4.6}$$

Remark 4.1. *Note that* [7] *used the notation* $S_{n,k}$ *for what we have called* $S_{k,n}$.

The random variable τ_k is the arrival time/index of the k^{th} pivot. The interval (L_k, R_k) gives the range of seeds to be compared to the k^{th} pivot in the operation of the `QuickVal` algorithm. The cost of comparing seed i to the k^{th} pivot is given by $X_{k,i}$. The total comparison costs attributed to the k^{th} pivot is $S_{k,n}$.

The cost of `QuickVal` on n keys is then given by

$$S_n := \sum_{k=1}^{\infty} S_{k,n}. \tag{4.7}$$

Define

$$\widehat{C}_K := \{C_k : k = 1, \ldots, K\},$$

and

$$\widehat{X}_{K,i} := \sum_{k=1}^{K} X_{k,i}.$$

Then, conditionally given $\widehat{C}_K$, the random variable

$$\widehat{S}_{K,n} := \sum_{\tau_K < i \leq n} \widehat{X}_{K,i}$$

is the sum of $(n - \tau_K)^+$ independent and identically distributed random variables, each with the same conditional distribution as $\widehat{X}_K := \sum_{k=1}^{K} X_k$, where

$$X_k := \mathbf{1}(L_{k-1} < U < R_{k-1})\beta(U, U_{\tau_k})$$

and U is uniformly distributed on $(0, 1)$ and independent of all the U_j's. Here, $\widehat{X}_{K,i}$ is the cost incurred by comparing seed i to pivots $1, 2, \ldots K$ and $\widehat{S}_{K,n}$ is the comparison cost of all seeds that arrive after the K-th pivot to pivots $1, 2, \ldots K$.

It will be helpful to condition on $\widehat{C}_K$ later. In anticipation of this, we establish notation for the conditional expectation of X_k given C_k (which equals the conditional expectation given $\widehat{C}_k$) and,

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for $k \le \ell$, the conditional expected product of X_k and X_ℓ given $\widehat{C}_\ell$, as follows:

$$I_k := \mathbb{E}[X_k | C_k] = \int_{L_{k-1}}^{R_{k-1}} \beta(u, U_{\tau_k})\, du, \tag{4.8}$$

$$I_{2,k,\ell} := \mathbb{E}[X_k X_\ell | \widehat{C}_\ell] = \int_{L_{\ell-1}}^{R_{\ell-1}} \beta(u, U_{\tau_k}) \beta(u, U_{\tau_\ell})\, du. \tag{4.9}$$

We symmetrize the definition of $I_{2,k,\ell}$ in the indices k and ℓ by setting $I_{2,k,\ell} := I_{2,\ell,k}$ for $k > \ell$. Finally, we write $I_{2,k}$ as shorthand for $I_{2,k,k}$.

We now calculate the mean and variance of $\widehat{X}_K$ with the intention of applying the classical central limit theorem; everything is done conditionally given $\widehat{C}_K$. Define μ_K and σ_K^2 to be the conditional mean and conditional variance of $\widehat{X}_K$ given $\widehat{C}_K$, respectively. Then

$$\mu_K = \sum_{k=1}^{K} I_k, \qquad \sigma_K^2 = \sum_{1 \le k, \ell \le K} \left(I_{2,k,\ell} - I_k I_\ell \right).$$

We next present a condition guaranteeing that σ_K^2 behaves well as $K \to \infty$. We note in passing that this condition is also the sufficient condition of Theorem 3.1 in [7] ensuring that S_n/n converges in L^2 to

$$S := \sum_{k \ge 1} I_k. \tag{4.10}$$

Lemma 4.2. *If*

$$\sum_{k=1}^{\infty} \left(\mathbb{E} I_{2,k} \right)^{1/2} < \infty, \tag{4.11}$$

then both almost surely and in L^1 we have that (i) the two series on the right in the equation

$$\sigma_\infty^2 := \sum_{k=1}^{\infty} (I_{2,k} - I_k^2) + 2 \sum_{\ell=1}^{\infty} \sum_{k=1}^{\ell-1} (I_{2,k,\ell} - I_k I_\ell) \tag{4.12}$$

converge absolutely, (ii) the equation holds, and (iii) $\sigma_K \xrightarrow{L^1} \sigma_\infty$ as $K \to \infty$.

Proof. Recall the notation $X_k = \mathbf{1}(L_{k-1} < U < R_{k-1}) \beta(U, U_{\tau_k})$ from above. Consider $1 \le k \le \ell$. The term $I_{2,k,\ell} - I_k I_\ell$ equals the conditional covariance of X_k and X_ℓ given $\widehat{C}_\ell$, and the absolute value of this conditional covariance is bounded above by the product of the conditional L^2-norms, namely, $I_{2,k}^{1/2} I_{2,\ell}^{1/2}$. Thus for the three desired conclusions it is sufficient that $\mathbb{E}\left(\sum_{k=1}^{\infty} I_{2,k}^{1/2} \right)^2 < \infty$. But

$$\mathbb{E} \left(\sum_{k=1}^{\infty} I_{2,k}^{1/2} \right)^2 = \left\| \sum_{k=1}^{\infty} I_{2,k}^{1/2} \right\|_2^2 \le \left(\sum_{k=1}^{\infty} \left\| I_{2,k}^{1/2} \right\|_2 \right)^2 = \left(\sum_{k=1}^{\infty} \left(\mathbb{E} I_{2,k} \right)^{1/2} \right)^2.$$

□

Remark 4.3. *In light of the absolute convergence noted in conclusion (i) of Lemma 4.2, we may unambiguously write*

$$\sigma_\infty^2 = \sum_{1 \le k, \ell < \infty} (I_{2,k,\ell} - I_k I_\ell), \tag{4.13}$$

both in L^1 and almost surely.

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Remark 4.4. *Note that if the source-specific cost function β is ϵ-tame for some $\epsilon < 1/2$, then, by Lemma 2.3 with $s = 2$ and $r = 1$, condition (4.11) in Lemma 4.2 is satisfied, because the series there enjoys geometric convergence.*

5. Convergence

Our main result is that, for a suitably tame cost function, the `QuickVal` residual converges in law to a scale-mixture of centered Gaussians. Furthermore, we have the explicit representation of Lemma 4.2 for the random scale σ_∞ as an infinite series of random variables that depend on conditional variances and covariances related to the source-specific cost functions [see (4.13) and (4.8)–(4.9)].

Theorem 5.1. *Suppose that the cost function β is ϵ-tame with $\epsilon < 1/2$. Then*

$$\sqrt{n}\left(\frac{S_n}{n} - S\right) \xrightarrow{\mathcal{L}} \sigma_\infty Z,$$

where Z has a standard normal distribution and is independent of σ_∞.

We approach the proof of Theorem 5.1 in two parts. First, in Proposition 5.4 we apply the central limit theorem to an approximation $\widehat{S}_{K,n}$ of the cost of `QuickVal` S_n. Second, we show that the error due to the approximation $\widehat{S}_{K,n}$ is negligible in the limit, culminating in the results of Propositions 5.9 and 5.11.

Before proving Theorem 5.1, we state a corollary to Theorem 5.1 for `QuickMin`. Recall that `QuickMin` is `QuickSelect` applied to find the minimum of the keys. Using a general source-specific cost function β, denote the cost of `QuickMin` on n keys by V_n. Since the operation of `QuickMin` is the same as that of `QuickVal` with $\alpha = 0$, Theorem 5.1 implies the following convergence for the cost of `QuickMin` with the same mild tameness condition on the source-specific cost function.

Corollary 5.2. *Suppose that the source-specific cost function β is ϵ-tame with $\epsilon < 1/2$. Then*

$$\sqrt{n}\left(\frac{V_n}{n} - S\right) \xrightarrow{\mathcal{L}} \sigma_\infty Z,$$

where Z has a standard normal distribution and is independent of σ_∞.

Remark 5.3. *In the key-comparisons case $\beta = 1$ (which is ϵ-tame for every $\epsilon \geq 0$) for $k \geq 0$ we have $L_k \equiv 0$ and $R_k \equiv U_{\tau_k}$, with the convention $U_{\tau_0} := 1$. Hence $I_k = U_{\tau_{k-1}}$ for $k \geq 1$, and $I_{2,k,\ell} = U_{\tau_{\ell-1}}$ for $1 \leq k \leq \ell$. Therefore $S = \sum_{k\geq 1} U_{\tau_{k-1}} = 1 + \sum_{k \geq 1} U_{\tau_k}$ and*

$$\sigma_\infty^2 = \sum_{1 \leq k, \ell < \infty} (1 - U_{\tau_k}) U_{\tau_\ell}$$

in Corollary 5.2. To further simplify the understanding of σ_∞^2, and hence of the limit in Corollary 5.2 in this case, observe that $U_{\tau_1}, U_{\tau_2}, \ldots$ have the same joint distribution as the cumulative products $U_1, U_1U_2, \ldots$. Thus

$$\sigma_\infty^2 \stackrel{\mathcal{L}}{=} \sum_{1 \leq k, \ell < \infty} \left[\left(1 - \prod_{i=1}^{k} U_i\right) \prod_{j=1}^{\ell} U_j\right].$$

Define

$$T_{K,n} := \frac{\widehat{S}_{K,n} - (n - \tau_K)^+ \mu_K}{\sqrt{n}}.$$

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Proposition 5.4. *Fix $K \in \{1, 2, \ldots\}$. Suppose that*

$$\mathbb{E}I_{2,k} < \infty$$

for $k = 1, 2, \ldots, K$. Then

$$T_{K,n} \xrightarrow{\mathcal{L}} \sigma_K Z$$

as $n \to \infty$, where Z has a standard normal distribution independent of σ_K.

Proof. The classical central limit theorem for independent and identically distributed random variables applied conditionally given $\widehat{C}_K$ yields

$$\mathcal{L}\left(\frac{\widehat{S}_{K,n} - (n-\tau_K)^+ \mu_K}{\sqrt{(n-\tau_K)^+}} \middle| \widehat{C}_K\right) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \sigma_K^2). \tag{5.1}$$

Since τ_K is finite almost surely, Slutsky's theorem (applied conditionally given $\widehat{C}_K$) implies that we can replace $\sqrt{(n-\tau_K)^+}$ by $\sqrt{n}$ in the denominator of (5.1). Finally, applying the dominated convergence theorem to conditional distribution functions gives that the resulting conditional convergence in distribution in (5.1) holds unconditionally. □

Define

$$W_{K,n} := \frac{1}{\sqrt{n}} \sum_{k=1}^{K} \sum_{\tau_k < i \le n} (X_{k,i} - I_k), \qquad \overline{W}_{K,n} := \frac{1}{\sqrt{n}} \sum_{k=K+1}^{\infty} \sum_{\tau_k < i \le n} (X_{k,i} - I_k),$$

and let

$$W_n := W_{K,n} + \overline{W}_{K,n}.$$

Note that W_n does not depend on K. We can write W_n in terms of the cost of `QuickVal` as follows:

$$W_n = \frac{1}{\sqrt{n}}\left(S_n - \sum_{k=1}^{\infty} (n-\tau_k)^+ I_k\right).$$

We prove that $W_n \xrightarrow{\mathcal{L}} \sigma_\infty Z$ (which is Proposition 5.9) in three parts. First (Lemma 5.5) we show that $\left|T_{K,n} - W_{K,n}\right| \to 0$ almost surely. Next (Lemma 5.7) we show that $\left\|\overline{W}_{K,n}\right\|_2$ is negligible as first $n \to \infty$ and then $K \to \infty$. Lastly (see the proof below of Proposition 5.9), an application of Markov's inequality gives the desired convergence.

Lemma 5.5. *For K fixed, if $\mathbb{E}I_k < \infty$ for $k = 1, 2, \ldots, K$, then*

$$\left|T_{K,n} - W_{K,n}\right| \to 0$$

almost surely as $n \to \infty$.

Remark 5.6. *The condition $\mathbb{E}I_k < \infty$ in Lemma 5.5 is weaker than the condition $\mathbb{E}I_{2,k} < \infty$ in Proposition 5.4.*

Proof of Lemma 5.5. When $n > \tau_K$ we have

$$\begin{aligned}\left|T_{K,n} - W_{K,n}\right| &= \frac{1}{\sqrt{n}}\left|\sum_{k=1}^{K} \sum_{\tau_k < i \le \tau_K} (X_{k,i} - I_k)\right| \\ &\le \frac{1}{\sqrt{n}} \sum_{k=1}^{K} \sum_{\tau_k < i \le \tau_K} \left|X_{k,i} - I_k\right|.\end{aligned}$$

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For a fixed K with $k \le K$, the almost sure finiteness of τ_k and τ_K implies that the sum

$$\sum_{k=1}^{K} \sum_{\tau_k < i \le \tau_K} \left| X_{k,i} - I_k \right|, \tag{5.2}$$

consists of an almost surely finite number of terms. Since each term $|X_{k,i} - I_k|$ is finite almost surely, the sum in (5.2) is finite almost surely. Therefore, $|T_{K,n} - W_{K,n}| \to 0$ almost surely as $n \to \infty$. □

Lemma 5.7. *Let*

$$\epsilon_K := \sum_{k=K+1}^{\infty} \left(\mathbb{E} I_{2,k}\right)^{1/2}.$$

Then

$$\left\| \overline{W}_{K,n} \right\|_2 \le \epsilon_K.$$

Remark 5.8. *A necessary and sufficient condition for $\epsilon_K \to 0$ as $K \to \infty$ is (4.11). Therefore, by Remark 4.4, ϵ-tameness for some $\epsilon < 1/2$ is sufficient.*

Proof of Lemma 5.7. Minkowski's inequality yields

$$\left\| \overline{W}_{K,n} \right\|_2 \le \frac{1}{\sqrt{n}} \sum_{k=K+1}^{\infty} \left\| \sum_{\tau_k < i \le n} (X_{k,i} - I_k) \right\|_2. \tag{5.3}$$

By conditioning on C_k, we can calculate the square of the L^2-norm here:

$$\begin{aligned} \left\| \sum_{\tau_k < i \le n} (X_{k,i} - I_k) \right\|_2^2 &= \mathbb{E}\, \mathbb{E}\left[\left(\sum_{\tau_k < i \le n} (X_{k,i} - I_k) \right)^2 \middle| C_k \right] \\ &= \mathbb{E}\left\{ (n - \tau_k)^+ (I_{2,k} - I_k^2) \right\} \\ &\le n \mathbb{E} I_{2,k}, \end{aligned} \tag{5.4}$$

where we use the fact that, conditionally given C_k, the random variables $X_{k,i} - I_k$ for $i > \tau_k$ are iid with zero mean. Substituting (5.4) into (5.3) gives the result. □

Proposition 5.9. *Suppose that*

$$\sum_{k=1}^{\infty} \left(\mathbb{E} I_{2,k}\right)^{1/2} < \infty.$$

Then

$$W_n \xrightarrow{\mathcal{L}} \sigma_\infty Z,$$

where Z has a standard normal distribution independent of σ_∞.

Proof. Let $t \in \mathbb{R}$ and $\delta > 0$. Since $W_n \le t$ implies either

$$W_{K,n} \le t + \delta \quad \text{or} \quad \left| W_n - W_{K,n} \right| > \delta,$$

we have

$$\mathbb{P}[W_n \le t] \le \mathbb{P}[W_{K,n} \le t + \delta] + \mathbb{P}[\,\left| W_n - W_{K,n} \right| > \delta]. \tag{5.5}$$

Markov's inequality and Lemma 5.7 imply

$$\mathbb{P}[\,\left| W_n - W_{K,n} \right| > \delta] \le \frac{\epsilon_K^2}{\delta^2}. \tag{5.6}$$

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Taking limits superior as $n \to \infty$ gives

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}[W_n \le t] &\le \limsup_{n\to\infty} \mathbb{P}[W_{K,n} \le t+\delta] + \frac{\epsilon_K^2}{\delta^2} \\ &\le \limsup_{n\to\infty} \mathbb{P}[T_{K,n} \le t+2\delta] + \frac{\epsilon_K^2}{\delta^2} \\ &= \mathbb{P}[\sigma_K Z \le t+2\delta] + \frac{\epsilon_K^2}{\delta^2},\end{aligned}$$

by (5.5)–(5.6), Lemma 5.5, and Proposition 5.4, respectively. Now taking limits as $K \to \infty$ gives

$$\limsup_{n\to\infty} \mathbb{P}[W_n \le t] \le \mathbb{P}[\sigma_\infty Z \le t+2\delta]$$

by Lemma 4.2 and the assumption that $\epsilon_K \to 0$. Letting $\delta \to 0$ yields

$$\limsup_{n\to\infty} \mathbb{P}[W_n \le t] \le \mathbb{P}[\sigma_\infty Z \le t]. \tag{5.7}$$

Applying the previous argument with limsup replaced by liminf to

$$\mathbb{P}[W_n \le t] \ge \mathbb{P}[W_{K,n} \le t-\delta] - \mathbb{P}[\,|W_n - W_{K,n}| \ge \delta]$$

implies

$$\liminf_{n\to\infty} \mathbb{P}[W_n \le t] \ge \mathbb{P}[\sigma_\infty Z < t]. \tag{5.8}$$

Since $\sigma_\infty Z$ has a continuous distribution, combining (5.7) and (5.8) gives the result. □

For completeness we include the following simple lemma, which will be needed in the sequel.

Lemma 5.10. *Let $0 < p < 1$ and $a_1, \ldots, a_n$ be non-negative real numbers. Then*

$$\left(\sum_{k=1}^{n} a_k\right)^p \le \sum_{k=1}^{n} a_k^p.$$

The final step in the proof of Theorem 5.1 is to show that the difference between the centering random variable

$$\sum_{k=1}^{\infty} (n-\tau_k)^+ I_k$$

in W_n and the more natural

$$nS = \sum_{k=1}^{\infty} nI_k$$

is negligible (when scaled by $1/\sqrt{n}$) in the limit as $n \to \infty$.

Proposition 5.11. *If the source-specific cost function β is ϵ-tame with $\epsilon < 1/2$, then*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\infty} \left[n - (n-\tau_k)^+\right] I_k \to 0$$

almost surely as $n \to \infty$.

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Proof. Observe that for any $0 < \delta < 1/2$, we have

$$[n - (n - \tau_k)^+] = \min(n, \tau_k) \le \tau_k^{(1/2)+\delta} n^{(1/2)-\delta}.$$

Therefore, if we let $0 < \delta < (1/2) - \epsilon$, it suffices to show that

$$\sum_{k=1}^{\infty} \tau_k^{(1/2)+\delta} I_k < \infty \tag{5.9}$$

almost surely. We prove this by showing that the random variable in (5.9) has finite expectation. Applying [7, Lemma 3.2] implies that for the ϵ-tameness constant c, we have

$$I_k \le \frac{2^\epsilon c}{1-\epsilon}(R_{k-1} - L_{k-1})^{1-\epsilon}.$$

Define, for $k = 1, 2, \ldots$, the sigma-field $\mathcal{F}_k := \sigma\langle (L_1, R_1), \ldots (L_{k-1}, R_{k-1})\rangle$. Conditionally given $\mathcal{F}_k$, the distribution of τ_k is the convolution over $j = 0, \ldots, k-1$ of geometric distributions with success probabilities $R_j - L_j$. This distribution is stochastically smaller than the convolution of k geometric distributions with success probability $R_{k-1} - L_{k-1}$. Let $G_k, G_{k,0}, \ldots, G_{k,k-1}$ be $k+1$ iid geometric random variables with success probability $R_{k-1} - L_{k-1}$. Then

$$\begin{aligned}
\mathbb{E}\left[\tau_k^{(1/2)+\delta} I_k \,\middle|\, \mathcal{F}_k\right] &\le C_1 \mathbb{E}\left[\left(\sum_{i=0}^{k-1} G_{k,i}\right)^{(1/2)+\delta} (R_{k-1} - L_{k-1})^{1-\epsilon} \,\middle|\, L_{k-1}, R_{k-1}\right] \\
&\le C_1 (R_{k-1} - L_{k-1})^{1-\epsilon}\, \mathbb{E}\left[\sum_{i=0}^{k-1} G_{k,i}^{(1/2)+\delta} \,\middle|\, L_{k-1}, R_{k-1}\right] \\
&\le C_1 k (R_{k-1} - L_{k-1})^{1-\epsilon}\, \mathbb{E}\left[G_k^{(1/2)+\delta} \,\middle|\, L_{k-1}, R_{k-1}\right],
\end{aligned} \tag{5.10}$$

where

$$C_1 := \frac{2^\epsilon c}{1-\epsilon}.$$

We can now compute

$$\mathbb{E}\left[G_k^{(1/2)+\delta} \,\middle|\, L_{k-1}, R_{k-1}\right] = \sum_{i=1}^{\infty} z^{i-1}(1-z) i^p, \tag{5.11}$$

where $z = 1 - (R_{k-1} - L_{k-1}) \in [0, 1)$ for $k \ge 2$ is the failure probability and $p = (1/2) + \delta$. Note that the infinite series in (5.11) can be written in terms of a polylogarithm function, as follows:

$$\sum_{i=1}^{\infty} z^{i-1}(1-z) i^p = z^{-1}(1-z)\mathrm{Li}_{-p,0}(z), \qquad \mathrm{Li}_{\alpha,r}(z) := \sum_{n=1}^{\infty} (\log i)^r \frac{z^i}{i^\alpha}.$$

Therefore [12, Theorem 1] implies the existence of an $\eta \in (0, 1)$ such that for $1 - \eta < z < 1$, we have

$$\sum_{i=1}^{\infty} z^i i^p \le \Gamma(1+p)(1-z)^{-(1+p)}.$$

On $0 \le z \le 1 - \eta$, the polylogarithm $\mathrm{Li}_{-p,0}(z)$ is increasing and therefore we have the bound

$$\mathrm{Li}_{-p,0}(z) \le \sum_{i=1}^{\infty} (1-\eta)^i i^p =: C_{p,\eta}$$

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Defining

$$C_2 := \max\left(\Gamma(1+p), C_{p,\eta}\right),$$

for $z \in [0, 1)$ we get

$$\mathrm{Li}_{-p,0}(z) \leq C_2(1-z)^{-(1+p)}. \tag{5.12}$$

Substituting the bound from (5.12) in (5.11) gives

$$\begin{aligned}\mathbb{E}\left[G_k^p | L_{k-1}, R_{k-1}\right] &\leq C_2 \frac{R_{k-1}-L_{k-1}}{1-(R_{k-1}-L_{k-1})}(R_{k-1}-L_{k-1})^{-(1+p)} \\ &= C_2 \sum_{j=0}^{\infty} (R_{k-1}-L_{k-1})^{j-p}.\end{aligned}$$

Therefore, after substituting $p = (1/2) + \delta$, an application of the monotone convergence theorem yields

$$\mathbb{E}(\tau_k^{(1/2)+\delta} I_k) \leq C_3 k \sum_{j=0}^{\infty} \mathbb{E}(R_{k-1}-L_{k-1})^{j+(1/2)-\epsilon-\delta},$$

where $C_3 := C_1 C_2$. Let $q := (1/2) - \epsilon - \delta$; then by the restriction placed on δ, we know $q > 0$. By [7, Lemma 3.1], we have

$$\mathbb{E}(\tau_k^{(1/2)+\delta} I_k) \leq C_3 k \sum_{j=0}^{\infty} \left(\frac{2-2^{-(j+q)}}{j+q+1}\right)^{k-1}.$$

Therefore, after defining

$$\gamma_j := \frac{2-2^{-(j+q)}}{j+q+1},$$

we have

$$\sum_{k=3}^{\infty} \mathbb{E}(\tau_k^{(1/2)+\delta} I_k) \leq C_3 \sum_{k=3}^{\infty} k \sum_{j=0}^{\infty} \gamma_j^{k-1} = C_3 \sum_{j=0}^{\infty} \sum_{k=3}^{\infty} k \gamma_j^{k-1} \leq 3C_3 \sum_{j=0}^{\infty} \frac{\gamma_j^2}{(1-\gamma_j)^2}.$$

Consequently, to check the convergence in (5.9), it suffices to check that $\sum_{j=0}^{\infty} \gamma_j^2 < \infty$; however, this follows trivially from the observation that $\gamma_j^2 \leq 4/j^2$. Therefore, it remains to show that the $k = 1$ and $k = 2$ terms in (5.9) have finite expectation. The first arrival time τ_1 equals 1 identically and $\mathbb{E}I_1 < \infty$. Applying (5.10) when $k = 2$ gives

$$\mathbb{E}\left[\tau_2^{(1/2)+\delta} I_2 \middle| \mathcal{F}_2\right] \leq 2C_1(R_1 - L_1)^{1-\epsilon}\, \mathbb{E}\left[G_2^{(1/2)+\delta} \middle| L_1, R_1\right].$$

Since $(R_1 - L_1)^{1-\epsilon} < 1$ a.s. , it suffices to show that

$$\mathbb{E}G_2^p < \infty \tag{5.13}$$

for $p = (1/2) + \delta$. However, we can calculate the expectation in (5.13) exactly. Since $R_1 - L_1 \overset{\mathcal{L}}{=} 1 - U$, where U has a unif(0, 1) distribution,

$$\mathbb{E}G_2^p = \sum_{i=1}^{\infty} i^p \mathbb{E}[(1-U)U^{i-1}] = \sum_{i=1}^{\infty} \frac{i^p}{i(i+1)},$$

which is finite because $p < 1$. □

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6. Convergence of moments for QuickVal residual

The main result of this section is that, under suitable tameness assumptions for the cost function, the moments of the normalized QuickVal residual converge to those of its limiting distribution.

Theorem 6.1. *Let $p \in [2, \infty)$. Suppose that the cost function β is ϵ-tame with $\epsilon < 1/p$. Then the moments of orders $\le p$ for the normalized QuickVal residual*

$$\sqrt{n}\left(\frac{S_n}{n} - S\right)$$

converge to the corresponding moments of the limit-law random variable $\sigma_\infty Z$.

Remark 6.2. *We will prove Theorem 6.1 using the second assertion in [2, Theorem 4.5.2]. Use of the first assertion in that theorem shows that, for all real $r \in [1, p]$, we also have convergence of rth absolute moments.*

As mentioned in Remark 6.2, we prove Theorem 6.1 using [2, Theorem 4.5.2] by proving that, for some $q > p$, the L^q-norms of the normalized QuickVal residuals are bounded in n. Choosing q arbitrarily from the nonempty interval $[2, 1/\epsilon)$ and using the triangle inequality for L^q-norm, we do this by showing (in Lemmas 6.3 and 6.4, respectively) that the same L^q-boundedness holds for each of the following two sequences:

$$\begin{aligned} W_n &= \frac{1}{\sqrt{n}}\left[S_n - \sum_{k=1}^{\infty}(n-\tau_k)^+ I_k\right] = \frac{1}{\sqrt{n}}\sum_{k=1}^{\infty}\left[S_{k,n} - (n-\tau_k)^+ I_k\right] \\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{\infty}\sum_{\tau_k < i \le n}(X_{k,i} - I_k), \end{aligned}$$

and the sequence previously treated in Proposition 5.11:

$$\widehat{W}_n := \frac{1}{\sqrt{n}}\sum_{k=1}^{\infty}\left[n - (n-\tau_k)^+\right] I_k.$$

Lemma 6.3. *Let $q \in [2, \infty)$, and suppose that the cost function β is ϵ-tame with $0 \le \epsilon < 1/q$. Then, the sequence (W_n) is L^q-bounded.*

Proof. This is straightforward. We proceed as at (5.3), except that we use triangle inequality for L^q-norm rather than for L^2-norm:

$$\|W_n\|_q \le \frac{1}{\sqrt{n}}\sum_{k=1}^{\infty}\left\|\sum_{\tau_k < i \le n}(X_{k,i} - I_k)\right\|_q .$$

To bound the L^q-norm on the right, we employ Rosenthal's inequality [27] conditionally given C_k to find

$$\begin{aligned} \left\|\sum_{\tau_k < i \le n}(X_{k,i} - I_k)\right\|_q^q &\le c_q\left[(n-\tau_k)^+\|X_k - I_k\|_q^q + \left((n-\tau_k)^+\right)^{q/2}\|X_k - I_k\|_2^2\right] \\ &\le c_q\left[n\|X_k - I_k\|_q^q + n^{q/2}\|X_k - I_k\|_2^2\right], \end{aligned}$$

and so, by Lemma 5.10,

$$\left\|\sum_{\tau_k < i \le n}(X_{k,i} - I_k)\right\|_q \le c_q^{1/q}\left[n^{1/q}\|X_k - I_k\|_q + n^{1/2}\|X_k - I_k\|_2^{2/q}\right].$$

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But by the argument at (5.4) we have

$$\|X_k - I_k\|_2^2 \le \mathbb{E} I_{2,k},$$

and

$$\|X_k - I_k\|_q \le \|X_k\|_q + \|I_k\|_q = \left(\mathbb{E} I_{q,k}\right)^{1/q} + \|I_k\|_q$$

by again conditioning on C_k to obtain the equality here. Consider a generalization of the definition of $I_{2,k} = I_{2,k,k}$ given in (4.9):

$$I_{q,k} := \mathbb{E}\left[X_k^q \,\middle|\, C_k \right] = \int_{L_{k-1}}^{R_{k-1}} \beta^q(u, U_{\tau_k})\, du.$$

Therefore

$$\left\| \sum_{\tau_k < i \le n} (X_{k,i} - I_k) \right\|_q \le c_q^{1/q} \left\{ n^{1/q} \left[\left(\mathbb{E} I_{q,k}\right)^{1/q} + \|I_k\|_q \right] + n^{1/2} \left(\mathbb{E} I_{2,k}\right)^{1/q} \right\}.$$

Three applications of Lemma 2.3 (requiring $\epsilon < 1/q$, $\epsilon < 1$, and $\epsilon < 1/2$ to handle $\mathbb{E} I_{q,k}$, $\|I_k\|_q$, and $\mathbb{E} I_{2,k}$, respectively), do the rest. □

Lemma 6.4. *Suppose that the cost function β is ϵ-tame with $0 \le \epsilon < 1/2$. Then, the sequence $(\widehat{W}_n)$ is L^q-bounded for every $q < \infty$.*

Proof. We may and do suppose $q \ge 2$. We begin as in the proof of Proposition 5.11, except that there is now no harm in choosing $\delta = 0$. So, it is sufficient to prove that

$$\sum_{k=1}^{\infty} \left\| \tau_k^{1/2} I_k \right\|_q < \infty.$$

We follow the proof of Proposition 5.11 to a large extent; in particular, what we will show is that all the terms in this sum are finite and that, for sufficiently large K, the series $\sum_{k=K}^{\infty}$ converges. As in the proof of Proposition 5.11, we utilise the bound

$$I_k \le \frac{2^\epsilon c}{1-\epsilon} (R_{k-1} - L_{k-1})^{1-\epsilon},$$

which requires only ϵ-tameness with $\epsilon < 1$. Then, we proceed much the same way as at (5.10), but now substituting convexity of qth power for use of Lemma 5.10:

$$\begin{aligned}
\mathbb{E}\left[\left(\tau_k^{1/2} I_k\right)^q \,\middle|\, \mathcal{F}_k \right] &\le C_1^q\, \mathbb{E}\left[\left(\sum_{i=0}^{k-1} G_{k,i} \right)^{q/2} (R_{k-1} - L_{k-1})^{q(1-\epsilon)} \,\middle|\, L_{k-1}, R_{k-1} \right] \\
&\le C_1^q\, (R_{k-1} - L_{k-1})^{q(1-\epsilon)} k^{(q/2)-1}\, \mathbb{E}\left[\sum_{i=0}^{k-1} G_{k,i}^{q/2} \,\middle|\, L_{k-1}, R_{k-1} \right] \\
&\le C_1^q k^{q/2} (R_{k-1} - L_{k-1})^{q(1-\epsilon)}\, \mathbb{E}\left[G_k^{q/2} \,\middle|\, L_{k-1}, R_{k-1} \right], \qquad (6.1)
\end{aligned}$$

where, as before, $C_1 = 2^\epsilon c/(1-\epsilon)$.

Arguing from here just as in the proof of Proposition 5.11, we find

$$\mathbb{E}\left[G_k^{q/2} \,\middle|\, L_{k-1}, R_{k-1} \right] \le C_2 \sum_{j=0}^{\infty} (R_{k-1} - L_{k-1})^{j-(q/2)}$$

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where $C_2 := \max(\Gamma(1+(q/2)), C_{q/2,\eta})$. (See the proof of Proposition 5.11 for the definition of $C_{q/2,\eta}$.) Therefore, with $C_3 := C_1^{q/2} C_2$, we have

$$\mathbb{E}\left[\left(\tau_k^{1/2} I_k\right)^q \,\middle|\, \mathcal{F}_k\right] \le C_3\, k^{q/2} \sum_{j=0}^{\infty} (R_{k-1} - L_{k-1})^{j+q(1-\epsilon)-(q/2)}.$$

By [7, Lemma 3.1], we have (using our assumption $\epsilon < 1/2$ for the $j=0$ term)

$$\mathbb{E}\left(\tau_k^{1/2} I_k\right)^q \le C_3\, k^{q/2} \sum_{j=0}^{\infty} \gamma_{j,q,\epsilon}^{k-1}, \tag{6.2}$$

where

$$\gamma_{j,q,\epsilon} := \frac{2 - 2^{-[j+q(1-\epsilon)-(q/2)]}}{j + q(1-\epsilon) - (q/2) + 1} \in (0,1)$$

decreases in j and vanishes in the limit as $j \to \infty$. Therefore, taking qth roots and using Lemma 5.10,

$$\left\| \tau_k^{1/2} I_k \right\|_q \le C_3^{q/2} k^{1/2} \sum_{j=0}^{\infty} \gamma_{j,q,\epsilon}^{(k-1)/q}.$$

If we bound the factor $k^{1/2}$ here by k and then sum the right side over $k \ge K$, the result is

$$C_3^{q/2} \sum_{j=0}^{\infty} \left[(K-1) \frac{\Gamma_j^{K-1}}{1-\Gamma_j} + \frac{\Gamma_j^{K-1}}{(1-\Gamma_j)^2} \right] \le C_3^{q/2} K \sum_{j=0}^{\infty} \frac{\Gamma_j^{K-1}}{(1-\Gamma_j)^2},$$

where

$$\Gamma_j \equiv \Gamma_{j,q,\epsilon} := \gamma_{j,q,\epsilon}^{1/q} \in (0,1),$$

like $\gamma_{j,q,\epsilon}$, decreases in j and vanishes in the limit as $j \to \infty$. Since $\Gamma_j < (2/j)^{1/q}$, it follows if we take $K \ge 2q+1$ that

$$\sum_{k=K}^{\infty} \left\| \tau_k^{1/2} I_k \right\|_q < \infty.$$

It remains to show that $\left\| \tau_k^{1/2} I_k \right\|_q < \infty$ for every k. For this we use (6.2) to note, since $0 < \gamma_{j,q,\epsilon} < 2/j$, that it clearly suffices to consider the cases $k=1$ and $k=2$. When $k=1$ we have $\tau_1 = 1$ and hence $\left\| \tau_1^{1/2} I_1 \right\|_q = \|I_1\|_q \le C_1 < \infty$. Applying (6.1) when $k=2$ gives

$$\mathbb{E}\left[\left(\tau_2^{1/2} I_2\right)^q \,\middle|\, \mathcal{F}_2\right] \le C_1^q 2^{q/2} (R_1 - L_1)^{q(1-\epsilon)}\, \mathbb{E}\left[G_2^{q/2} \,\middle|\, L_1, R_1\right],$$

and we can exactly compute

$$\begin{aligned}
&\mathbb{E}\left\{(R_1 - L_1)^{q(1-\epsilon)}\, \mathbb{E}\left[G_2^{q/2} \,\middle|\, L_1, R_1\right]\right\} \\
&\quad = \mathbb{E}\left\{(R_1 - L_1)^{q(1-\epsilon)} \sum_{i=1}^{\infty} i^{q/2} (R_1 - L_1)[1 - (R_1 - L_1)]^{i-1}\right\} \\
&\quad = \sum_{i=1}^{\infty} i^{q/2} \mathbb{E}\left[U^{i-1}(1-U)^{q(1-\epsilon)+1}\right] = \sum_{i=1}^{\infty} i^{q/2} B(i, q(1-\epsilon)+2)
\end{aligned}$$

where $U \sim \text{unif}(0, 1)$. Each of the terms in this last sum is finite, and by Stirling's formula the ith term equals $(1 + o(1))i^{-[2+((1/2)-\epsilon)q]} = o(i^{-2})$ as $i \to \infty$, so the sum converges. Hence $\|\tau_2^{1/2} I_2\|_q < \infty$. □

Remark 6.5. *Matterer [25 , Chapter 7] describes the approach, involving the contraction method for inspiration and the method of moments for proof, we initially took in trying to establish a limiting distribution for the* `QuickVal` *residual in the special case of* `QuickMin` *with key-comparisons cost. It turns out that, for this approach, we must consider the* `QuickMin` *limit and the residual from it bivariately. However, we discovered that, unfortunately, the limit residual* `QuickMin` *distribution is not uniquely determined by its moments (we omit the proof here); so, the method of moments approach is ultimately unsuccessful, unlike for* `QuickSort` [13]. *We nevertheless find that approach instructive, since it does yield a rather direct proof of convergence of moments for the residual in the special case of* `QuickMin` *with key-comparisons cost; see [25 , Chapter 7] for details.*

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Competing interests

The authors declare none.

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